A Minimal Prikry-type Forcing for Singularizing a Measurable Cardinal

Karen Räsch
University of Münster

Hejnice, 03. February 2011

Introduction

- This talk introduces a minimal Prikry-type forcing, i.e., it has the typical properties of Prikry-type forcings while every generic extension by it has no proper intermediate models.
- There are lots of minimal forcings, like Sacks forcing or Laver forcing. Other forcings such as Cohen forcing are not minimal.
- It is known that the classical Prikry forcing and also the supercompact Prikry-type forcing have many intermediate models.

Introduction

- This talk introduces a minimal Prikry-type forcing, i.e., it has the typical properties of Prikry-type forcings while every generic extension by it has no proper intermediate models.
- There are lots of minimal forcings, like Sacks forcing or Laver forcing. Other forcings such as Cohen forcing are not minimal.
- It is known that the classical Prikry forcing and also the supercompact Prikry-type forcing have many intermediate models.

Introduction

- This talk introduces a minimal Prikry-type forcing, i.e., it has the typical properties of Prikry-type forcings while every generic extension by it has no proper intermediate models.
- There are lots of minimal forcings, like Sacks forcing or Laver forcing. Other forcings such as Cohen forcing are not minimal.
- It is known that the classical Prikry forcing and also the supercompact Prikry-type forcing have many intermediate models.

Generic Extensions by the Classical Prikry Forcing

This work was inspired by the following result

Theorem (Gitik, Kanovei, Koepke, 2010)

Let $V[G]$ be a generic extension by classical Prikry forcing.

Then every intermediate model is a Prikry extension by this forcing and is generated by some subsequence of the associated Prikry sequence.

Moreover, the intermediate models of V and $V[G]$ ordered by inclusion are isomorphic to $\mathscr{P}(\omega)/\text{finite}$ ordered by almost inclusion.

Generic Extensions by the Classical Prikry Forcing

This work was inspired by the following result

Theorem (Gitik, Kanovei, Koepke, 2010)

Let $V[G]$ be a generic extension by classical Prikry forcing.
Then every intermediate model is a Prikry extension by this forcing and is generated by some subsequence of the associated Prikry sequence.
Moreover, the intermediate models of V and $V[G]$ ordered by inclusion are isomorphic to $\mathscr{P}(\omega)/\text{finite}$ ordered by almost inclusion.

Generic Extensions by the Classical Prikry Forcing

This work was inspired by the following result

Theorem (Gitik, Kanovei, Koepke, 2010)

Let $V[G]$ be a generic extension by classical Prikry forcing.

Then every intermediate model is a Prikry extension by this forcing and is generated by some subsequence of the associated Prikry sequence.

Moreover, the intermediate models of V and $V[G]$ ordered by inclusion are isomorphic to $\mathscr{P}(\omega)/\mathrm{finite}$ ordered by almost inclusion.

Generic Extensions by the Classical Prikry Forcing

This work was inspired by the following result

Theorem (Gitik, Kanovei, Koepke, 2010)

Let $V[G]$ be a generic extension by classical Prikry forcing.

Then every intermediate model is a Prikry extension by this forcing and is generated by some subsequence of the associated Prikry sequence.

Moreover, the intermediate models of V and $V[G]$ ordered by inclusion are isomorphic to $\mathscr{P}(\omega)/\text{finite}$ ordered by almost inclusion.

Outline

For the rest of the talk let κ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $\mathbb{P}_{\mathcal{U}}$.

We obtain a standard Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ.

Then for every $X \in V[G]$ either $X \in V$ or X generates the whole generic extension, i.e., $V[X] = V[G]$.

Outline

For the rest of the talk let κ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $\mathbb{P}_{\mathcal{U}}$.

We obtain a standard Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ.

Then for every $X \in V[G]$ either $X \in V$ or X generates the whole generic extension, i.e., $V[X] = V[G]$.

Outline

For the rest of the talk let κ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $\mathbb{P}_{\mathcal{U}}$.

We obtain a standard Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ.

Then for every $X \in V[G]$ either $X \in V$ or X generates the whole generic extension, i.e., $V[X] = V[G]$.

Outline

For the rest of the talk let κ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $\mathbb{P}_{\mathcal{U}}$.

We obtain a standard Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ.

Then for every $X \in V[G]$ either $X \in V$ or X generates the whole generic extension, i.e., $V[X] = V[G]$.

Outline

For the rest of the talk let κ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $\mathbb{P}_{\mathcal{U}}$.

We obtain a standard Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ.

Then for every $X \in V[G]$ either $X \in V$ or X generates the whole generic extension, i.e., $V[X] = V[G]$.

Outline

For the rest of the talk let κ be a measurable cardinal.

The classical Prikry forcing is equivalent to a Prikry tree forcing. Inspired by this, we define the partial order $\mathbb{P}_{\mathcal{U}}$.

We obtain a standard Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which makes it worthy of being called a Prikry-type forcing.

The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ.

Then for every $X \in V[G]$ either $X \in V$ or X generates the whole generic extension, i.e., $V[X] = V[G]$.

Preliminaries

- Think of $u, v \in [\kappa]^{<\omega}$ as strictly increasing sequences of ordinals. By $u \trianglelefteq v$ we mean that u is an initial segment of v. Concatenation is denoted by the symbol $^\frown$; the restriction of the domain by $\restriction$.
- A tree is a non-empty subset of $[\kappa]^{<\omega}$ which is closed under initial segments. $\mathrm{Lev}_k(T)$ denotes the k-th level of T.
- We denote the minimal inner model of ZFC containing the set $X \subseteq V$ and incorporating V by $V[X]$.

  We say X is V-constructibly equivalent to Y, in short $X \equiv_V Y$, if $V[X] = V[Y]$.

Preliminaries

- Think of $u, v \in [\kappa]^{<\omega}$ as strictly increasing sequences of ordinals. By $u \trianglelefteq v$ we mean that u is an initial segment of v. Concatenation is denoted by the symbol $^\frown$; the restriction of the domain by $\restriction$.
- A tree is a non-empty subset of $[\kappa]^{<\omega}$ which is closed under initial segments. $\mathrm{Lev}_k(T)$ denotes the k-th level of T.
- We denote the minimal inner model of ZFC containing the set $X \subseteq V$ and incorporating V by $V[X]$.

  We say X is V-constructibly equivalent to Y, in short $X \equiv_V Y$, if $V[X] = V[Y]$.

Preliminaries

- Think of $u, v \in [\kappa]^{<\omega}$ as strictly increasing sequences of ordinals. By $u \trianglelefteq v$ we mean that u is an initial segment of v. Concatenation is denoted by the symbol $^\frown$; the restriction of the domain by $\restriction$.
- A tree is a non-empty subset of $[\kappa]^{<\omega}$ which is closed under initial segments. $\mathrm{Lev}_k(T)$ denotes the k-th level of T.
- We denote the minimal inner model of ZFC containing the set $X \subseteq V$ and incorporating V by $V[X]$.

  We say X is V-constructibly equivalent to Y, in short $X \equiv_V Y$, if $V[X] = V[Y]$.

Preliminaries

- Think of $u, v \in [\kappa]^{<\omega}$ as strictly increasing sequences of ordinals. By $u \trianglelefteq v$ we mean that u is an initial segment of v. Concatenation is denoted by the symbol $^\frown$; the restriction of the domain by $\upharpoonright$.
- A tree is a non-empty subset of $[\kappa]^{<\omega}$ which is closed under initial segments. $\mathrm{Lev}_k(T)$ denotes the k-th level of T.
- We denote the minimal inner model of ZFC containing the set $X \subseteq V$ and incorporating V by $V[X]$.

  We say X is V-constructibly equivalent to Y, in short $X \equiv_V Y$, if $V[X] = V[Y]$.

Preliminaries

- We will not deal with inner models of ZFC but with sets of ordinals.

  Reason: Every intermediate inner model $V \subseteq M \subseteq V[G]$ of ZFC is generated by a single set. Hence consider all sets of ordinals in $V[G]$ with the equivalence relation $\equiv_V$.

- Fix a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on κ.

  The consistency strength is

  $$\mathrm{ZFC} + \text{“there exists a measurable cardinal”}$$

  by a theorem of Kunen and Paris.

  For the minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Preliminaries

- We will not deal with inner models of ZFC but with sets of ordinals.

  Reason: Every intermediate inner model $V \subseteq M \subseteq V[G]$ of ZFC is generated by a single set. Hence consider all sets of ordinals in $V[G]$ with the equivalence relation $\equiv_V$.

- Fix a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on κ.

  The consistency strength is

  $$\mathrm{ZFC} + \text{“there exists a measurable cardinal”}$$

  by a theorem of Kunen and Paris.

  For the minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Preliminaries

- We will not deal with inner models of ZFC but with sets of ordinals.

  Reason: Every intermediate inner model $V \subseteq M \subseteq V[G]$ of ZFC is generated by a single set. Hence consider all sets of ordinals in $V[G]$ with the equivalence relation $\equiv_V$.

- Fix a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on κ.

  The consistency strength is

  $$\mathrm{ZFC} + \text{“there exists a measurable cardinal”}$$

  by a theorem of Kunen and Paris.

  For the minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Preliminaries

- We will not deal with inner models of ZFC but with sets of ordinals.

  Reason: Every intermediate inner model $V \subseteq M \subseteq V[G]$ of ZFC is generated by a single set. Hence consider all sets of ordinals in $V[G]$ with the equivalence relation $\equiv_V$.

- Fix a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on κ.

  The consistency strength is

  ZFC + "there exists a measurable cardinal"

  by a theorem of Kunen and Paris.

  For the minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Preliminaries

- We will not deal with inner models of ZFC but with sets of ordinals.

  Reason: Every intermediate inner model $V \subseteq M \subseteq V[G]$ of ZFC is generated by a single set. Hence consider all sets of ordinals in $V[G]$ with the equivalence relation $\equiv_V$.

- Fix a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on κ.

  The consistency strength is

  ZFC + "there exists a measurable cardinal"

  by a theorem of Kunen and Paris.

  For the minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Definition of a $\mathcal{U}$-Tree

Definition

A set $T \subseteq [\kappa]^{<\omega}$ is called $\mathcal{U}$-tree with trunk t if

- $\langle T, \trianglelefteq \rangle$ is a tree.
- $t \in T$ and for all $u \in T$ we have $u \trianglelefteq t$ or $t \trianglelefteq u$.
- For all $u \in T$ if $t \trianglelefteq u$ then

$$\mathrm{Suc}_T(u) := \{ \xi < \kappa \ : \ u^\frown\langle \xi \rangle \in T \} \in U_{\max(u)}.$$

Definition of a $\mathcal{U}$-Tree

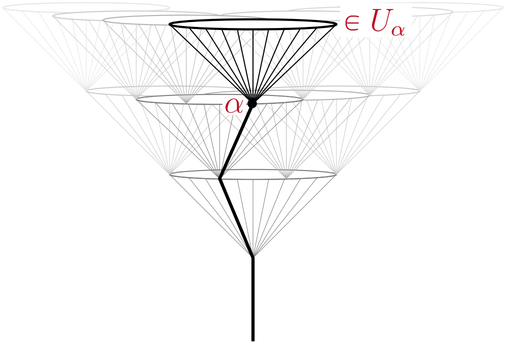
Definition

A set $T \subseteq [\kappa]^{<\omega}$ is called $\mathcal{U}$-tree with trunk t if

- $\langle T, \trianglelefteq \rangle$ is a tree.
- $t \in T$ and for all $u \in T$ we have $u \trianglelefteq t$ or $t \trianglelefteq u$.
- For all $u \in T$ if $t \trianglelefteq u$ then

$$\mathrm{Suc}_T(u) := \{ \xi < \kappa \ : \ u^\frown\langle \xi \rangle \in T \} \ \in \ U_{\max(u)}.$$

An Image of a $\mathcal{U}$-Tree



Some Properties of $\mathcal{U}$-Trees

- Let $u \in T$, $u \trianglerighteq t$. Then
$$T \upharpoonright u := \{ v \in T \ : \ u \trianglelefteq v \vee v \trianglelefteq u \}$$
is a $\mathcal{U}$-tree with trunk u and $\langle t, T \rangle \geqslant \langle u, T \upharpoonright u \rangle$.
- The intersection of less than κ many $\mathcal{U}$-trees all having the same trunk t is again a $\mathcal{U}$-tree with trunk t.

Some Properties of $\mathcal{U}$-Trees

- Let $u \in T$, $u \trianglerighteq t$. Then
$$T \restriction u := \{ v \in T \ : \ u \trianglelefteq v \vee v \trianglelefteq u \}$$
is a $\mathcal{U}$-tree with trunk u and $\langle t, T \rangle \geqslant \langle u, T \restriction u \rangle$.

- The intersection of less than κ many $\mathcal{U}$-trees all having the same trunk t is again a $\mathcal{U}$-tree with trunk t.

Some Properties of $\mathcal{U}$-Trees

- Let $u \in T$, $u \trianglerighteq t$. Then
$$T \upharpoonright u := \{ v \in T \ : \ u \trianglelefteq v \lor v \trianglelefteq u \}$$
is a $\mathcal{U}$-tree with trunk u and $\langle t, T \rangle \geqslant \langle u, T \upharpoonright u \rangle$.

- The intersection of less than κ many $\mathcal{U}$-trees all having the same trunk t is again a $\mathcal{U}$-tree with trunk t.

The Partial Order $\mathbb{P}_{\mathcal{U}}$

Definition

Let $\mathbb{P}_{\mathcal{U}} := \{ \langle t, T \rangle \ : \ T$ is a $\mathcal{U}$-tree with trunk $t \}$.

Furthermore for $\langle s, S \rangle, \langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$ define

$$\langle s, S \rangle \leqslant \langle t, T \rangle \quad \text{if} \quad S \subseteq T$$

$$\langle s, S \rangle \leqslant^* \langle t, T \rangle \quad \text{if} \quad S \subseteq T \text{ and } s = t$$

The Partial Order $\mathbb{P}_{\mathcal{U}}$

Definition

Let $\mathbb{P}_{\mathcal{U}} := \{ \langle t, T \rangle \ : \ T$ is a $\mathcal{U}$-tree with trunk $t \}$.

Furthermore for $\langle s, S \rangle, \langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$ define

$$\langle s, S \rangle \leqslant \langle t, T \rangle \quad \text{if} \quad S \subseteq T$$
$$\langle s, S \rangle \leqslant^* \langle t, T \rangle \quad \text{if} \quad S \subseteq T \text{ and } s = t$$

The Partial Order $\mathbb{P}_\mathcal{U}$

Definition

Let $\mathbb{P}_\mathcal{U} := \{\langle t, T\rangle : T$ is a $\mathcal{U}$-tree with trunk $t\}$.

Furthermore for $\langle s, S\rangle, \langle t, T\rangle \in \mathbb{P}_\mathcal{U}$ define

$$\langle s, S\rangle \leqslant \langle t, T\rangle \quad \text{if} \quad S \subseteq T$$

$$\langle s, S\rangle \leqslant^* \langle t, T\rangle \quad \text{if} \quad S \subseteq T \text{ and } s = t$$

Colorings of $\mathcal{U}$-Trees

Lemma (Colorings of $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T \to \lambda$ with $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk homogeneous for c, i.e., every two elements of c on the same level get the same color.

Proof.

Straightforward induction. □

Colorings of $\mathcal{U}$-Trees

Lemma (Colorings of $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T \to \lambda$ with $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk homogeneous for c, i.e., every two elements of c on the same level get the same color.

Proof.

Straightforward induction. □

Colorings of $\mathcal{U}$-Trees

Lemma (Colorings of $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T \to \lambda$ with $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk homogeneous for c, i.e., every two elements of c on the same level get the same color.

Proof.

Straightforward induction. □

Colorings of $\mathcal{U}$-Trees

Lemma (Colorings of $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T \to \lambda$ with $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk homogeneous for c, i.e., every two elements of c on the same level get the same color.

Proof.

Straightforward induction. □

Graphs on $\mathcal{U}$-Trees

Definition

For $u, v \in [\kappa]^{<\omega}$ enumerate $u \cup v$ strictly increasing as $\{\xi_i : i < n\}$ and define $\mathrm{type}(u, v) \in 3^n$ by

$$\mathrm{type}(u,v)(i) = \begin{cases} 0 & \text{if } \xi_i \in u \setminus v \\ 1 & \text{if } \xi_i \in v \setminus u \\ 2 & \text{if } \xi_i \in u \cap v. \end{cases}$$

Graphs on $\mathcal{U}$-Trees

Definition

For $u, v \in [\kappa]^{<\omega}$ enumerate $u \cup v$ strictly increasing as $\{ \xi_i : i < n \}$ and define $\mathrm{type}(u, v) \in 3^n$ by

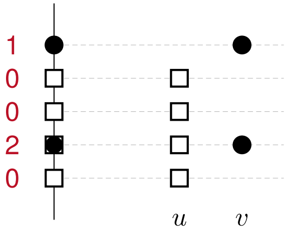
$$\mathrm{type}(u,v)(i) = \begin{cases} 0 & \text{if } \xi_i \in u \setminus v \\ 1 & \text{if } \xi_i \in v \setminus u \\ 2 & \text{if } \xi_i \in u \cap v. \end{cases}$$

Graphs on $\mathcal{U}$-Trees

Definition

For $u, v \in [\kappa]^{<\omega}$ enumerate $u \cup v$ strictly increasing as $\{ \xi_i : i < n \}$ and define $\mathrm{type}(u, v) \in 3^n$ by

$$\mathrm{type}(u,v)(i) = \begin{cases} 0 & \text{if } \xi_i \in u \setminus v \\ 1 & \text{if } \xi_i \in v \setminus u \\ 2 & \text{if } \xi_i \in u \cap v. \end{cases}$$



Graphs on $\mathcal{U}$-Trees

Lemma (Graphs on $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T^2 \to \lambda$ for some $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in \bar{T}$ the value of c only depends on the type of u, v.

Proof.

First use the normality to define a diagonal intersection of $\mathcal{U}$-trees. The proof itself is a quite technical induction with lots of case distinctions. □

Graphs on $\mathcal{U}$-Trees

Lemma (Graphs on $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T^2 \to \lambda$ for some $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in \bar{T}$ the value of c only depends on the type of u, v.

Proof.

First use the normality to define a diagonal intersection of $\mathcal{U}$-trees. The proof itself is a quite technical induction with lots of case distinctions. □

Graphs on $\mathcal{U}$-Trees

Lemma (Graphs on $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T^2 \to \lambda$ for some $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in \bar{T}$ the value of c only depends on the type of u, v.

Proof.

First use the normality to define a diagonal intersection of $\mathcal{U}$-trees. The proof itself is a quite technical induction with lots of case distinctions. □

Forcing with $\mathbb{P}_{\mathcal{U}}$

- Classical Prikry tree forcing is $\mathbb{P}_{\mathcal{U}}$ when all U_α equal the same κ-complete nonprincipal ultrafilter U over κ.
- Let G be generic on $\mathbb{P}_{\mathcal{U}}$. As usual
$$f_G := \bigcup \{ t : \exists T \, \langle t, T \rangle \in G \}$$
is an ω-sequence cofinal in κ, called Prikry sequence.
- G consists of all $\mathcal{U}$-trees of which f_G is a branch, i.e., $f_G \equiv_V G$.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant \rangle$ satisfies the κ^+-cc.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant^* \rangle$ is κ-closed.

Forcing with $\mathbb{P}_{\mathcal{U}}$

- Classical Prikry tree forcing is $\mathbb{P}_{\mathcal{U}}$ when all U_α equal the same κ-complete nonprincipal ultrafilter U over κ.
- Let G be generic on $\mathbb{P}_{\mathcal{U}}$. As usual
$$f_G := \bigcup \{ t : \exists T \langle t, T \rangle \in G \}$$
is an ω-sequence cofinal in κ, called Prikry sequence.
- G consists of all $\mathcal{U}$-trees of which f_G is a branch, i.e., $f_G \equiv_V G$.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant \rangle$ satisfies the κ^+-cc.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant^* \rangle$ is κ-closed.

Forcing with $\mathbb{P}_{\mathcal{U}}$

- Classical Prikry tree forcing is $\mathbb{P}_{\mathcal{U}}$ when all U_α equal the same κ-complete nonprincipal ultrafilter U over κ.
- Let G be generic on $\mathbb{P}_{\mathcal{U}}$. As usual
$$f_G := \bigcup \{ t : \exists T \langle t, T \rangle \in G \}$$
is an ω-sequence cofinal in κ, called Prikry sequence.
- G consists of all $\mathcal{U}$-trees of which f_G is a branch, i.e., $f_G \equiv_V G$.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant \rangle$ satisfies the κ^+-cc.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant^* \rangle$ is κ-closed.

Forcing with $\mathbb{P}_{\mathcal{U}}$

- Classical Prikry tree forcing is $\mathbb{P}_{\mathcal{U}}$ when all U_α equal the same κ-complete nonprincipal ultrafilter U over κ.
- Let G be generic on $\mathbb{P}_{\mathcal{U}}$. As usual
$$f_G := \bigcup \{ t : \exists T \, \langle t, T \rangle \in G \}$$
is an ω-sequence cofinal in κ, called Prikry sequence.
- G consists of all $\mathcal{U}$-trees of which f_G is a branch, i.e., $f_G \equiv_V G$.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant \rangle$ satisfies the κ^+-cc.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant^* \rangle$ is κ-closed.

Forcing with $\mathbb{P}_{\mathcal{U}}$

- Classical Prikry tree forcing is $\mathbb{P}_{\mathcal{U}}$ when all U_α equal the same κ-complete nonprincipal ultrafilter U over κ.
- Let G be generic on $\mathbb{P}_{\mathcal{U}}$. As usual
$$f_G := \bigcup \{ t : \exists T \, \langle t, T \rangle \in G \}$$
is an ω-sequence cofinal in κ, called Prikry sequence.
- G consists of all $\mathcal{U}$-trees of which f_G is a branch, i.e., $f_G \equiv_V G$.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant \rangle$ satisfies the κ^+-cc.
- $\langle \mathbb{P}_{\mathcal{U}}, \leqslant^* \rangle$ is κ-closed.

Forcing with $\mathbb{P}_{\mathcal{U}}$

Lemma (Prikry lemma)

Let $\langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$ and φ a statement in the forcing language.
Then there is a direct extension $\langle s, S\rangle \in \mathbb{P}_{\mathcal{U}}$ of $\langle t, T\rangle$ deciding φ.

Forcing with $\mathbb{P}_{\mathcal{U}}$

Lemma (Prikry lemma)

Let $\langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$ and φ a statement in the forcing language. Then there is a direct extension $\langle s, S \rangle \in \mathbb{P}_{\mathcal{U}}$ of $\langle t, T \rangle$ deciding φ.

Proof.

Follows easily from the lemma about colorings of $\mathcal{U}$-trees. □

Forcing with $\mathbb{P}_{\mathcal{U}}$

Lemma (Prikry lemma)

Let $\langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$ and φ a statement in the forcing language. Then there is a direct extension $\langle s, S\rangle \in \mathbb{P}_{\mathcal{U}}$ of $\langle t, T\rangle$ deciding φ.

The following theorem sums up what we achieved so far.

Theorem

Let G be a generic filter on $\mathbb{P}_{\mathcal{U}}$. Then in $V[G]$

- *κ is singular with $\mathrm{cf}(\kappa) = \aleph_0$.*
- *No bounded subsets of κ are added.*
- *All cardinals are preserved and also all cofinalities but κ's.*

The Theorem

Remember:

- $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ is a sequence of pairwise distinct normal measures on the measurable cardinal κ.
- $\langle A_\alpha : \alpha < \kappa \rangle$ is a family of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$.

The Theorem

Remember:

- $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ is a sequence of pairwise distinct normal measures on the measurable cardinal κ.
- $\langle A_\alpha : \alpha < \kappa \rangle$ is a family of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$.

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Proof of the Theorem

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$.

The proof splits into two parts:

Part I . Subsets of κ in $V[G]$

Part II. Arbitrary sets of ordinals in $V[G]$

Proof of the Theorem

Theorem (Koepke, Schlicht, R., 2010)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ either $X \in V$ or $X \equiv_V f_G$.

The proof splits into two parts:

Part I . Subsets of κ in $V[G]$

Part II. Arbitrary sets of ordinals in $V[G]$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees

Lemma (Graphs on $\mathcal{U}$-trees)

Let T be a $\mathcal{U}$-tree and $c : T^2 \to \lambda$, $\lambda < \kappa$.

Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that the value of c only depends on the type of the arguments.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees and

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees and

Lemma

Let T be a $\mathcal{U}$-tree. Then there is $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in T$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ in both domains.

Proof.

Simply restrict $\mathrm{Suc}_T(u)$ to $A_{\max(u)}$ for all $u \in T$. □

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

We will use the lemma about graphs on $\mathcal{U}$-trees and

Lemma

Let T be a $\mathcal{U}$-tree. Then there is $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in T$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ in both domains.

Proof.

Simply restrict $\mathrm{Suc}_T(u)$ to $A_{\max(u)}$ for all $u \in T$. □

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$.

Goal: Find $p \leqslant \langle t, T \rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T \upharpoonright u \rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{ \xi < \max(u) : \langle u, T \upharpoonright u \rangle \Vdash \check{\xi} \in \dot{X} \}$.
Consider $c : T \times T \to 2$, where

$$\langle u, v \rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$.

Goal: Find $p \leqslant \langle t, T \rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T \restriction u \rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{ \xi < \max(u) : \langle u, T \restriction u \rangle \Vdash \check{\xi} \in \dot{X} \}$.

Consider $c : T \times T \to 2$, where

$$\langle u, v \rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$.

Goal: Find $p \leqslant \langle t, T\rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T\restriction u\rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{\xi < \max(u) : \langle u, T\restriction u\rangle \Vdash \check{\xi} \in \dot{X}\}$.

Consider $c : T \times T \to 2$, where

$$\langle u, v\rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$.

Goal: Find $p \leqslant \langle t, T\rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T\restriction u\rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{\xi < \max(u) : \langle u, T\restriction u\rangle \Vdash \check{\xi} \in \dot{X}\}$.

Consider $c : T \times T \to 2$, where

$$\langle u, v\rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T\rangle \in \mathbb{P}_{\mathcal{U}}$.

Goal: Find $p \leqslant \langle t, T\rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T \upharpoonright u\rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{\xi < \max(u) : \langle u, T \upharpoonright u\rangle \Vdash \check{\xi} \in \dot{X}\}$.

Consider $c : T \times T \to 2$, where

$$\langle u, v\rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof.

Let $\dot{X}$ be a name for some $X \subseteq \kappa$ and $\langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$.

Goal: Find $p \leqslant \langle t, T \rangle$ such that $p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$.

By the Prikry lemma assume that for all $u \in T$ the condition $\langle u, T \restriction u \rangle$ already decides $\dot{X}$ up to $\max(u)$.

For $u \in T$ define $X_u := \{ \xi < \max(u) : \langle u, T \restriction u \rangle \Vdash \check{\xi} \in \dot{X} \}$.

Consider $c : T \times T \to 2$, where

$$\langle u, v \rangle \mapsto 1 \quad \text{iff} \quad X_u \cap \max(v) = X_v \cap \max(u).$$

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Thin out T and obtain $\bar{T} \subseteq T$ such that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type
- for all $u, v \in \bar{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ in both domains.

Claim 1. Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof. Later!

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Thin out T and obtain $\bar{T} \subseteq T$ such that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type
- for all $u, v \in \bar{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ in both domains.

Claim 1. Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof. Later!

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Thin out T and obtain $\bar{T} \subseteq T$ such that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type
- for all $u, v \in \bar{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ in both domains.

Claim 1. Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof. Later!

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Thin out T and obtain $\bar{T} \subseteq T$ such that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type
- for all $u, v \in \bar{T}$ with $u(n) \neq v(n)$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ in both domains.

Claim 1. Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\upharpoonright s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\upharpoonright s) : u(|s|) \neq v(|s|)\}.$$

Proof. Later!

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. Let G be generic, $\langle t, \bar{T} \rangle \in G$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T}\rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v\rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.
Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.
Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T}\rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v\rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 0.

Then all $v \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.

Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T}\rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. There is $n > 0$ such that the only value of c on $\{\langle u, v\rangle \in \mathrm{Lev}_{m+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{m+n}(\bar{T}\restriction s) : u(m) \neq v(m)\}$ is 0.
Then all $v \in \mathrm{Lev}_{m+n}(\bar{T}\restriction s)$ with $v(m) \neq f_G(m)$ satisfy $c(v, f_G \restriction (m+n)) = 0$.
Hence $X_v \neq \dot{X}^G \cap \max(v)$. This uniquely determines $f_G(m)$.

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T}\rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. ✓

Case 2. For all $n > 0$ the only value of c on $\{\langle u, v\rangle \in \mathrm{Lev}_{m+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{m+n}(\bar{T}\restriction s) : u(m) \neq v(m)\}$ is 1.

Then $X_{s^\frown\langle\xi\rangle} = \dot{X}^G \cap \xi$ for all ξ, i.e., $\dot{X}^G = \bigcup_{\xi \in \mathrm{Suc}_{\bar{T}}(s)} X_{s^\frown\langle\xi\rangle} \in V$. □

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. ✓

Case 2. For all $n > 0$ the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 1.

Then $X_{s^\frown\langle \xi \rangle} = \dot{X}^G \cap \xi$ for all ξ, i.e., $\dot{X}^G = \bigcup_{\xi \in \mathrm{Suc}_{\bar{T}}(s)} X_{s^\frown\langle \xi \rangle} \in V$. □

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. ✓

Case 2. For all $n > 0$ the only value of c on
$\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 1.

Then $X_{s^\frown\langle \xi \rangle} = \dot{X}^G \cap \xi$ for all ξ, i.e., $\dot{X}^G = \bigcup_{\xi \in \mathrm{Suc}_{\bar{T}}(s)} X_{s^\frown\langle \xi \rangle} \in V$. □

Proof of the Theorem – Part I

Theorem (Part I)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \subseteq \kappa$ in $V[G]$ either $X \in V$ or $X \equiv_V f_G$.

Sketch of the proof (continued).

Claim 2. $\langle t, \bar{T} \rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Proof. We have $X_{f_G \restriction (k+1)} = \dot{X}^G \cap f_G(k)$ for all k. Assume $\dot{X}^G \notin V$.

How to construct f_G from $\dot{X}^G$: Assume we know $s := f_G \restriction m$.

Case 1. ✓

Case 2. For all $n > 0$ the only value of c on $\{\langle u, v \rangle \in \mathrm{Lev}_{m+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{m+n}(\bar{T} \restriction s) : u(m) \neq v(m)\}$ is 1.

Then $X_{s^\frown \langle \xi \rangle} = \dot{X}^G \cap \xi$ for all ξ, i.e., $\dot{X}^G = \bigcup_{\xi \in \mathrm{Suc}_{\bar{T}}(s)} X_{s^\frown \langle \xi \rangle} \in V$. □

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T} \restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T} \upharpoonright s) \times \mathrm{Lev}_{|s|+n}(\bar{T} \upharpoonright s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T} \restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\upharpoonright s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\upharpoonright s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T} \upharpoonright s) \times \mathrm{Lev}_{|s|+n}(\bar{T} \upharpoonright s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1.

First remember that

- the values of c on $\bar{T} \times \bar{T}$ only depend on the type,
- $u(m) \neq v(m)$ for all $\langle u, v\rangle$ in the above set, all $|s| \leqslant m < |s| + n$.

If there are $\langle u, v\rangle$ in the above set with $c(u, v) = 1$,
then for all $\langle u', v'\rangle$ in the above set $c(u', v') = 1$.

Three steps to see this:

$$\mathrm{type}(u, v) \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \mathrm{type}(u', v')$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1,0,0,1,0,1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1,0,0,0,1,1\rangle$$

Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1,0,0,1,0,1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1,0,0,0,1,1\rangle$$

Proof of the Theorem – Part I

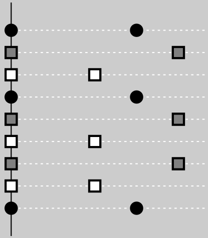
Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1, 0, 0, 1, 0, 1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1, 0, 0, 0, 1, 1\rangle$$



Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T} \restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T} \restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1, 0, 0, 1, 0, 1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1, 0, 0, 0, 1, 1\rangle$$

Proof of the Theorem – Part I

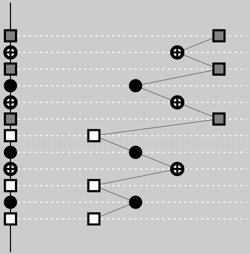
Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T} \upharpoonright s) \times \mathrm{Lev}_{|s|+n}(\bar{T} \upharpoonright s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1,0,0,1,0,1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1,0,0,0,1,1\rangle$$



Proof of the Theorem – Part I

Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1,0,0,1,0,1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1,0,0,0,1,1\rangle$$

Proof of the Theorem – Part I

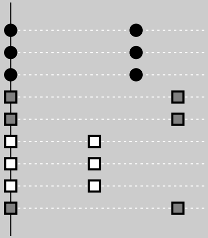
Claim 1

Let $s \in \bar{T}$ and $n < \omega$. Then c is constant on the set

$$\{\langle u, v\rangle \in \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) \times \mathrm{Lev}_{|s|+n}(\bar{T}\restriction s) : u(|s|) \neq v(|s|)\}.$$

Proof of Claim 1 (continued) for $n = 3$.

$$\langle 1,0,0,1,0,1\rangle \rightsquigarrow \mathbb{t}_{alternating} \rightsquigarrow \mathbb{t}_{successive} \rightsquigarrow \langle 1,0,0,0,1,1\rangle$$



□

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$.

We may assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ of ordinals in γ.

Fix a sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc we can code $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from Y_ξ in some $Y \subseteq \kappa$.

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$.

We may assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ of ordinals in γ.

Fix a sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc we can code $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from Y_ξ in some $Y \subseteq \kappa$.

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$.

We may assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ of ordinals in γ.

Fix a sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc we can code $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from Y_ξ in some $Y \subseteq \kappa$.

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$.

We may assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ of ordinals in γ.

Fix a sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc we can code $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from Y_ξ in some $Y \subseteq \kappa$.

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$.

We may assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ of ordinals in γ.

Fix a sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc we can code $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from Y_ξ in some $Y \subseteq \kappa$.

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$.

We may assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ of ordinals in γ.

Fix a sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ in $V[X]$ such that $Y_\xi \equiv_V X \cap \gamma_\xi$.

Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc we can code $\langle Y_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ and a recipe to obtain $X \cap \gamma_\xi$ from Y_ξ in some $Y \subseteq \kappa$.

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.

Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$. ✓

Case 2. $\mathrm{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. □

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$. ✓

Case 2. $\mathrm{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. □

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$. ✓

Case 2. $\mathrm{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. □

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$. ✓

Case 2. $\mathrm{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. □

Proof of the Theorem – Part II

Theorem (Part II)

Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$.
Then for every $X \in V[G]$ there exists $Y \subseteq \kappa$ in $V[G]$ with $X \equiv_V Y$.

Proof.

Proceed by induction on the least γ with $X \subseteq \gamma$. Assume $\gamma > \kappa$.

Case 1. $\mathrm{cf}(\gamma) \leqslant \kappa$. ✓

Case 2. $\mathrm{cf}(\gamma) > \kappa$.

By the induction hypothesis either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ for every $\xi < \gamma$.

We may assume that $X \cap \xi \in V$ for all $\xi < \gamma$.

Show: If $X \cap \xi \in V$ for all $\xi < \gamma$, then $X \in V$. □

Further Remarks

Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of κ.

Further Remarks

Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of κ.

Further Remarks

Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of κ.

Further Remarks

Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of κ.

Further Remarks

Now drop the assumption of normality. Then

- we still have a Prikry-type forcing,
- this forcing will not be minimal in general because of the Covering Theorem for $L[U]$,
- it is still possible to reduce the problem to subsets of κ.

Thanks for listening! ☺